

ON k -FIBONACCI SUMS BY MATRIX METHODS

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ABSTRACT. In this paper, some k -Fibonacci and k -Lucas with arithmetic indexes sums are derived by using the matrices $R_a = \begin{bmatrix} L_{k,a} & -(-1)^a \\ 1 & 0 \end{bmatrix}$ and $S_a = \frac{1}{2} \begin{bmatrix} L_{k,a} & \Delta_a \\ 1 & L_{k,a} \end{bmatrix}$, where $\Delta_a = L_{k,a}^2 - 4(-1)^a$.

The most notable side of this paper is our proof method, since all the identities used in the proofs of main theorems are proved previously by using the matrices R_a and S_a , with $a \in \mathbb{N}$. Although the identities we proved are known, our proofs are not encountered in the k -Fibonacci and k -Lucas numbers literature.

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1. INTRODUCTION

One of the more studied sequences is the Fibonacci sequence [1], and it has been generalized in many ways [2, 3]. Here, we use the following one-parameter generalization of the Fibonacci sequence.

Definition 1.1. For any integer number $k \geq 1$, the k -th Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by

$$(1) \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1,$$

where $F_{k,0} = 0$ and $F_{k,1} = 1$.

Note that for $k = 1$ the classical Fibonacci sequence is obtained while for $k = 2$ we obtain the Pell sequence. Some of the properties that the k -Fibonacci numbers verify and that we will need later are summarized below [4]:

[Binet's formula] $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$, where $\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$. These roots verify $\sigma_1 + \sigma_2 = k$ and $\sigma_1 \sigma_2 = -1$.

This paper presents an interesting investigation about some special relations between matrices and k -Fibonacci and k -Lucas numbers. This investigation is valuable, since it provides students to use their theoretical knowledge to obtain new k -Fibonacci and k -Lucas identities with arithmetic indexes by different methods.

We focus here on the subsequences of k -Fibonacci numbers with indexes in an arithmetic sequence, say $an + r$ for fixed integers a, r with $0 \leq r \leq a - 1$. Several formulas for the sums of such numbers are deduced by matrix methods.

2. MAIN THEOREMS

Let us denote $F_{k,n+1} + F_{k,n-1}$ by $L_{k,n}$ (the k -Lucas numbers).

Theorem 2.1. *Let a be a fixed positive integer. If T is a square matrix with $T^2 = L_{k,a}T - (-1)^a I$ and I the matrix identity of order 2. Then,*

$$(2) \quad T^n = \frac{1}{F_{k,a}} (F_{k,an}T - (-1)^a F_{k,a(n-1)}I),$$

for all $n \in \mathbb{Z}$.

Proof. If $n = 0$, the proof is obvious because $F_{k,-a} = -(-1)^a F_{k,a}$. It can be shown by induction that $F_{k,a}T^n = F_{k,an}T - (-1)^a F_{k,a(n-1)}I$, for every positive integer n . We now show that

$$(3) \quad T^{-n} = \frac{1}{F_{k,a}} (F_{k,a(-n)}T - (-1)^a F_{k,a(-n-1)}I).$$

Let $U = L_{k,a}I - T = (-1)^a T^{-1}$, then

$$\begin{aligned} U^2 &= (L_{k,a}I - T)^2 = L_{k,a}^2 I - 2L_{k,a}T + T^2 \\ &= L_{k,a}(L_{k,a}I - T) - (-1)^a I = L_{k,a}U - (-1)^a I, \end{aligned}$$

this shows that $U^n = \frac{1}{F_{k,a}} (F_{k,an}U - (-1)^a F_{k,a(n-1)}I)$.

That is, $F_{k,a}((-1)^a T^{-1})^n = F_{k,an}(L_{k,a}I - T) - (-1)^a F_{k,a(n-1)}I$. Therefore

$$\begin{aligned} (-1)^{an}(F_{k,a}T^{-n}) &= -F_{k,an}T + (L_{k,a}F_{k,an} - (-1)^a F_{k,a(n-1)})I \\ &= -F_{k,an}T + F_{k,a(n+1)}I. \end{aligned}$$

Thus,

$$(4) \quad T^{-n} = \frac{1}{F_{k,a}} (-(-1)^{-an} F_{k,an}T + (-1)^{-an} F_{k,a(n+1)}I).$$

Thus, the proof is completed. \square

Now, we define a 2×2 matrix R_a and then we give some new results for the k -Fibonacci numbers $F_{k,an}$ by matrix methods.

Define the 2×2 matrix R_a as follows:

$$(5) \quad R_a = \begin{bmatrix} L_{k,a} & -(-1)^a \\ 1 & 0 \end{bmatrix}.$$

By an inductive argument and using (2), we get

Corollary 2.2. *For any integer $n \geq 1$ holds:*

$$(6) \quad R_a^n = \frac{1}{F_{k,a}} \begin{bmatrix} F_{k,a(n+1)} & -(-1)^a F_{k,an} \\ F_{k,an} & -(-1)^a F_{k,a(n-1)} \end{bmatrix}.$$

Clearly the matrix R_a^n satisfies the recurrence relation, for $n \geq 1$

$$(7) \quad R_a^{n+1} = L_{k,a} R_a^n - (-1)^a R_a^{n-1},$$

where $R_a^0 = I$ and $R_a^1 = R_a$.

We define S_a be the 2×2 matrix

$$(8) \quad S_a = \frac{1}{2} \begin{bmatrix} L_{k,a} & \Delta_a \\ 1 & L_{k,a} \end{bmatrix},$$

where $\Delta_a = L_{k,a}^2 - 4(-1)^a$. Then,

Corollary 2.3. *For any integer $n \geq 1$ holds:*

$$(9) \quad S_a^n = \frac{1}{2F_{k,a}} \begin{bmatrix} \epsilon_a(n) & \Delta_a F_{k,an} \\ F_{k,an} & \epsilon_a(n) \end{bmatrix}.$$

where $\epsilon_a(n) = 2F_{k,a(n+1)} - L_{k,a}F_{k,an}$.

Proof. (By induction). For $n = 1$:

$$(10) \quad S_a^1 = \frac{1}{2} \begin{bmatrix} L_{k,a} & \Delta_a \\ 1 & L_{k,a} \end{bmatrix} = \frac{1}{2F_{k,a}} \begin{bmatrix} F_{k,a}L_{k,a} & \Delta_a F_{k,a} \\ F_{k,a} & F_{k,a}L_{k,a} \end{bmatrix}$$

since $\epsilon_a(1) = F_{k,2a} = L_{k,a}F_{k,a}$. Let us suppose that the formula is true for $n-1$:

$$(11) \quad S_a^{n-1} = \frac{1}{2F_{k,a}} \begin{bmatrix} \epsilon_a(n-1) & \Delta_a F_{k,a(n-1)} \\ F_{k,a(n-1)} & \epsilon_a(n-1) \end{bmatrix},$$

with $\epsilon_a(n-1) = 2F_{k,an} - L_{k,a}F_{k,a(n-1)}$.

Then,

$$\begin{aligned} S_a^n &= S_a^{n-1} S_a^1 = \frac{1}{4F_{k,a}} \begin{bmatrix} \epsilon_a(n-1) & \Delta_a F_{k,a(n-1)} \\ F_{k,a(n-1)} & \epsilon_a(n-1) \end{bmatrix} \begin{bmatrix} L_{k,a} & \Delta_a \\ 1 & L_{k,a} \end{bmatrix} \\ &= \frac{1}{4F_{k,a}} \begin{bmatrix} \epsilon_a(n-1)L_{k,a} + \Delta_a F_{k,a(n-1)} & \Delta_a(\epsilon_a(n-1) + L_{k,a}F_{k,a(n-1)}) \\ \epsilon_a(n-1) + L_{k,a}F_{k,a(n-1)} & \epsilon_a(n-1)L_{k,a} + \Delta_a F_{k,a(n-1)} \end{bmatrix} \\ &= \frac{1}{2F_{k,a}} \begin{bmatrix} \epsilon_a(n) & \Delta_a F_{k,an} \\ F_{k,an} & \epsilon_a(n) \end{bmatrix}, \end{aligned}$$

since $2\epsilon_a(n) = \epsilon_a(n-1)L_{k,a} + \Delta_a F_{k,a(n-1)}$. Thus, the proof is completed. \square

An important property of these numbers can be tested using the above result.

Theorem 2.4. *For any integer $n \geq 1$ holds:*

$$(12) \quad F_{k,a(n+1)}^2 - L_{k,a}F_{k,an}F_{k,a(n+1)} + (-1)^a F_{k,an}^2 = F_{k,a}^2 (-1)^{an}.$$

Proof. Since $\det(S_a) = (-1)^a$, $\det(S_a^n) = (\det(S_a))^n = (-1)^{an}$. Moreover, since (9), we get $\det(S_a^n) = \frac{1}{4F_{k,a}^2}(\epsilon_a(n)^2 - \Delta_a F_{k,an}^2)$. Furthermore,

$$\epsilon_a(n)^2 - \Delta_a F_{k,an}^2 = 4(F_{k,a(n+1)}^2 - L_{k,a}F_{k,an}F_{k,a(n+1)} + (-1)^a F_{k,an}^2).$$

The proof is completed. \square

Let us give a different proof of one of the fundamental identities of k -Fibonacci and k -Lucas numbers, by using the matrix S_a .

Theorem 2.5. *For all $n, m \in \mathbb{N}$,*

$$(13) \quad F_{k,a}F_{k,a(n+m)} = F_{k,a(n+1)}F_{k,am} + F_{k,a(m+1)}F_{k,an} - L_{k,a}F_{k,an}F_{k,am}.$$

Proof. Since $S_a^{n+m} = S_a^n S_a^m$, then

$$\begin{bmatrix} \epsilon_a(n+m) & \Delta_a F_{k,a(n+m)} \\ F_{k,a(n+m)} & \epsilon_a(n+m) \end{bmatrix} = \frac{1}{2F_{k,a}} \begin{bmatrix} \epsilon_a(n) & \Delta_a F_{k,an} \\ F_{k,an} & \epsilon_a(n) \end{bmatrix} \begin{bmatrix} \epsilon_a(m) & \Delta_a F_{k,am} \\ F_{k,am} & \epsilon_a(m) \end{bmatrix},$$

where $\epsilon_a(n) = 2F_{k,a(n+1)} - L_{k,a}F_{k,an}$. It is seen that,

$$(14) \quad 2F_{k,a}S_a^{n+m} = \begin{bmatrix} \epsilon_a(n)\epsilon_a(m) + \Delta_a F_{k,an}F_{k,am} & \Delta_a(\epsilon_a(m)F_{k,an} + \epsilon_a(n)F_{k,am}) \\ \epsilon_a(m)F_{k,an} + \epsilon_a(n)F_{k,am} & \epsilon_a(n)\epsilon_a(m) + \Delta_a F_{k,an}F_{k,am} \end{bmatrix}.$$

Thus it follows that,

$$(15) \quad 2F_{k,a}F_{k,a(n+m)} = \epsilon_a(m)F_{k,an} + \epsilon_a(n)F_{k,am},$$

and

$$\epsilon_a(m)F_{k,an} + \epsilon_a(n)F_{k,am} = 2(F_{k,a(n+1)}F_{k,am} + F_{k,a(m+1)}F_{k,an} - L_{k,a}F_{k,an}F_{k,am}).$$

Then, the proof is completed. \square

In the particular case, if $a = 1$, we obtain

Corollary 2.6. *For all $n, m \in \mathbb{N}$,*

$$(16) \quad F_{k,n+m} = F_{k,m+1}F_{k,n} + F_{k,m}F_{k,n-1}.$$

Theorem 2.7. *For all $n, m \in \mathbb{N}$,*

$$(17) \quad (-1)^{am}F_{k,a}F_{k,a(n-m)} = F_{k,a(m+1)}F_{k,an} - F_{k,a(n+1)}F_{k,am}.$$

Proof. Since $S_a^{n-m} = S_a^n (S_a^m)^{-1}$, then

$$\begin{bmatrix} \epsilon_a(n-m) & \Delta_a F_{k,a(n-m)} \\ F_{k,a(n-m)} & \epsilon_a(n-m) \end{bmatrix} = \frac{(-1)^{am}}{2F_{k,a}} \begin{bmatrix} \epsilon_a(n) & \Delta_a F_{k,an} \\ F_{k,an} & \epsilon_a(n) \end{bmatrix} \begin{bmatrix} \epsilon_a(m) & -\Delta_a F_{k,am} \\ -F_{k,am} & \epsilon_a(m) \end{bmatrix},$$

where $\epsilon_a(n) = 2F_{k,a(n+1)} - L_{k,a}F_{k,an}$. It is seen that,

$$2(-1)^{am} F_{k,a} S_a^{n-m} = \begin{bmatrix} \epsilon_a(n)\epsilon_a(m) - \Delta_a F_{k,an}F_{k,am} & \Delta_a(\epsilon_a(m)F_{k,an} - \epsilon_a(n)F_{k,am}) \\ \epsilon_a(m)F_{k,an} - \epsilon_a(n)F_{k,am} & \epsilon_a(n)\epsilon_a(m) - \Delta_a F_{k,an}F_{k,am} \end{bmatrix}.$$

Thus it follows that,

$$(19) \quad 2(-1)^{am} F_{k,a} F_{k,a(n-m)} = \epsilon_a(m)F_{k,an} - \epsilon_a(n)F_{k,am},$$

and

$$\epsilon_a(m)F_{k,an} - \epsilon_a(n)F_{k,am} = 2(F_{k,a(m+1)}F_{k,an} - F_{k,a(n+1)}F_{k,am}).$$

Then, the proof is completed. \square

In the particular case, if $a = 1$, we obtain

Corollary 2.8. *For all $n, m \in \mathbb{N}$,*

$$(20) \quad (-1)^m F_{k,n-m} = F_{k,m+1}F_{k,n} - F_{k,n+1}F_{k,m}.$$

3. SUM OF k -FIBONACCI NUMBERS OF KIND an

In this section, we study the sum of the k -Fibonacci numbers of kind an , with a an positive integer number.

Theorem 3.1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $a \geq 1$. Then,*

$$(21) \quad \sum_{i=0}^n F_{k,ai} = \frac{(-1)^a F_{k,an} + F_{k,a} - F_{k,a(n+1)}}{1 + (-1)^a - L_{k,a}}.$$

Proof. It is known that $I - S_a^{n+1} = (I - S_a) \sum_{i=0}^n S_a^i$. If $\det(I - S_a)$ is nonzero, then we can write

$$(22) \quad (I - S_a)^{-1} (I - S_a^{n+1}) = \sum_{i=0}^n S_a^i = \frac{1}{2F_{k,a}} \begin{bmatrix} \sum_{i=0}^n \epsilon_a(i) & \Delta_a \sum_{i=0}^n F_{k,ai} \\ \sum_{i=0}^n F_{k,ai} & \sum_{i=0}^n \epsilon_a(i) \end{bmatrix}.$$

where $\epsilon_a(i) = 2F_{k,a(i+1)} - L_{k,a}F_{k,ai}$.

It is easy to see that,

$$(23) \quad \det(I - S_a) = \left(1 - \frac{1}{2}L_{k,a}\right)^2 - \frac{1}{4}\Delta_a = 1 + (-1)^a - L_{k,a}$$

is nonzero, because $a \geq 1$. If we take $\delta = 1 + (-1)^a - L_{k,a}$, then we get

$$(24) \quad (I - S_a)^{-1} = \frac{1}{\delta} \begin{bmatrix} 1 - \frac{1}{2}L_{k,a} & \frac{\Delta_a}{2} \\ \frac{1}{2} & 1 - \frac{1}{2}L_{k,a} \end{bmatrix} = \frac{1}{\delta} \left[\left(1 - \frac{1}{2}L_{k,a}\right) I + \frac{1}{2}T_a \right],$$

where $T_a = \begin{bmatrix} 0 & \Delta_a \\ 1 & 0 \end{bmatrix}$.

Thus it is seen that,

$$\begin{aligned} (I - S_a)^{-1}(I - S_a^{n+1}) &= \frac{1}{\delta} \left[\left(1 - \frac{1}{2}L_{k,a}\right) I + \frac{1}{2}T_a \right] (I - S_a^{n+1}) \\ &= \frac{1}{\delta} \left[\left(1 - \frac{1}{2}L_{k,a}\right) (I - S_a^{n+1}) + \frac{1}{2}T_a(I - S_a^{n+1}) \right], \end{aligned}$$

where

$$T_a(I - S_a^{n+1}) = \frac{1}{2F_{k,a}} \begin{bmatrix} -\Delta_a F_{k,a(n+1)} & \Delta_a(2F_{k,a} - \epsilon_a(n+1)) \\ 2F_{k,a} - \epsilon_a(n+1) & -\Delta_a F_{k,a(n+1)} \end{bmatrix}.$$

Furthermore, from the identity (22), it follows that

$$\begin{aligned} \sum_{i=0}^n F_{k,ai} &= \frac{1}{\delta} \left(- \left(1 - \frac{1}{2}L_{k,a}\right) F_{k,a(n+1)} + \frac{1}{2}(2F_{k,a} - \epsilon_a(n+1)) \right) \\ &= \frac{1}{\delta} ((-1)^a F_{k,an} + F_{k,a} - F_{k,a(n+1)}). \end{aligned}$$

□

Theorem 3.2. *Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $a \geq 1$. Then,*

$$(25) \quad \sum_{i=0}^n (-1)^i F_{k,ai} = \frac{(-1)^a F_{k,an} - F_{k,a} + F_{k,a(n+1)}}{1 + (-1)^a + L_{k,a}}.$$

Proof. We prove the theorem in two phases, by taking n as an even and odd natural number. Firstly assume that n is an even natural number. Then,

$$I + S_a^{n+1} = (I + S_a) \sum_{i=0}^n (-1)^i S_a^i.$$

If $\det(I + S_a)$ is nonzero, then we can write

$$(26) \quad (I + S_a)^{-1}(I + S_a^{n+1}) = \sum_{i=0}^n S_a^i = \frac{1}{2F_{k,a}} \begin{bmatrix} \sum_{i=0}^n (-1)^i \epsilon_a(i) & \Delta_a \sum_{i=0}^n (-1)^i F_{k,ai} \\ \sum_{i=0}^n (-1)^i F_{k,ai} & \sum_{i=0}^n (-1)^i \epsilon_a(i) \end{bmatrix}.$$

where $\epsilon_a(i) = 2F_{k,a(i+1)} - L_{k,a}F_{k,ai}$.

It is easy to see that,

$$(27) \quad \det(I + S_a) = \left(1 + \frac{1}{2}L_{k,a}\right)^2 - \frac{1}{4}\Delta_a = 1 + (-1)^a + L_{k,a}$$

is nonzero. If we take $\delta = 1 + (-1)^a + L_{k,a}$, then we get

$$(28) \quad (I + S_a)^{-1} = \frac{1}{\delta} \begin{bmatrix} 1 + \frac{1}{2}L_{k,a} & -\frac{\Delta_a}{2} \\ -\frac{1}{2} & 1 + \frac{1}{2}L_{k,a} \end{bmatrix} = \frac{1}{\delta} \left[\left(1 + \frac{1}{2}L_{k,a}\right) I - \frac{1}{2}T_a \right],$$

where $T_a = \begin{bmatrix} 0 & \Delta_a \\ 1 & 0 \end{bmatrix}$.

Thus it is seen that,

$$\begin{aligned} (I + S_a)^{-1}(I + S_a^{n+1}) &= \frac{1}{\delta} \left[\left(1 + \frac{1}{2}L_{k,a} \right) I - \frac{1}{2}T_a \right] (I + S_a^{n+1}) \\ &= \frac{1}{\delta} \left[\left(1 + \frac{1}{2}L_{k,a} \right) (I + S_a^{n+1}) - \frac{1}{2}T_a(I + S_a^{n+1}) \right], \end{aligned}$$

where

$$T_a(I + S_a^{n+1}) = \frac{1}{2F_{k,a}} \begin{bmatrix} \Delta_a F_{k,a(n+1)} & \Delta_a(2F_{k,a} + \epsilon_a(n+1)) \\ 2F_{k,a} + \epsilon_a(n+1) & \Delta_a F_{k,a(n+1)} \end{bmatrix}.$$

Furthermore, from the identity (22), it follows that

$$\begin{aligned} \sum_{i=0}^n (-1)^i F_{k,ai} &= \frac{1}{\delta} \left(\left(1 + \frac{1}{2}L_{k,a} \right) F_{k,a(n+1)} + \frac{1}{2}(2F_{k,a} + \epsilon_a(n+1)) \right) \\ &= \frac{1}{\delta} ((-1)^a F_{k,an} - F_{k,a} + F_{k,a(n+1)}). \end{aligned}$$

Now assume that n is an odd natural number. Hence we get,

$$(29) \quad \sum_{i=0}^n (-1)^i F_{k,ai} = \sum_{i=0}^{n-1} (-1)^i F_{k,ai} - F_{k,an}$$

Since n is an odd natural number, then $n-1$ is even. Thus taking $n-1$ in (25) and using it in (29), the proof is completed. \square

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